

Representability, amalgamation in connection to the finitizability problem for Heyting polyadic algebras

The Finitizability problem for predicate intuitionistic logic

Tarek Sayed Ahmed

April 8, 2013

Abstract

We prove completeness and interpolation for infinitary predicates intuitionistic logic, by proving new neat embedding theorems for Heyting polyadic algebras. Also we formulate and solve the analogue of the lengthy discussed finitizability problem in (classical) algebraic logic, which has to do with finding finitely based varieties adequate for the algebraisation of predicate intuitionistic logic circumventing non-finite axiomatizability results.

1

We follow standard notation. G will always be a semigroup of transformations on a set or ordinal (well ordered set) under the operation of composition. All undefined terms, concepts in this part of the paper which we use extensively without warning are found in Part 1. Cross references will be given.

This part is organized as follows. In section 2 we give a simple proofs of representation theorems for the countable case, using ideas of Ono. In section 2, we prove various representation theorems in the presence of diagonal elements.

1 Simpler proofs

Throughout G is a semigroup. We start by giving a simpler proof of our representability results when G consists only of finite transformations and G is

¹ 2000 *Mathematics Subject Classification*. Primary 03G15.

Key words: algebraic logic, neat reducts, cylindric algebras, amalgamation

strongly rich. We assume that everything is countable; algebras, dimensions and semigroups. We do not study the case when G consists of all transformations since in this case we have uncountably many operations (even if the dimension is countable).

Dimensions will be specified by countable ordinals for some time to come. An ω *dilation* of an algebra of dimension α is its minimal dilation of dimension $\alpha + \omega$. Our next proof is a typical Henkin construction that does not involve zigzagging. It is an algebraic version of a proof of Ono, but it proves much more.

We treat the two cases simultaneously; V denotes the class of algebras and $\mathfrak{F}\mathfrak{r}_\beta^\rho V$ denotes the free countable V algebra, where ρ is not dimension restricting in the second case while $\alpha \sim \rho(i)$ is infinite in the first case [cf. definition ? in part 1].

- Definition 1.1.** (1) Let \mathfrak{A} be an algebra. Let $\Gamma, \Delta \subseteq \mathfrak{A}$. We write $\Gamma \rightarrow \Delta$, if there exist $n, m \in \omega$, $a_1, \dots, a_n \in \Gamma$, $b_1, \dots, b_m \in \Delta$ such that $a_1 \wedge \dots \wedge a_n \leq b_1 \vee \dots \vee b_m$. Recall that a pair (Γ, Δ) is consistent if not $\Gamma \rightarrow \Delta$.
- (2) We say that a theory (Γ, Δ) is complete, if it is consistent and $\Gamma \cup \Delta = \mathfrak{A}$.
- (3) A theory (Γ, Δ) is Henkin complete in \mathfrak{A} if it is complete and saturated. $\Gamma \subseteq A$ is saturated, if there exists (equivalently for all) $\Delta \subseteq A$, such that (Γ, Δ) is saturated.

Definition 1.2. (1) Let $\Gamma \subseteq \mathfrak{S}g^\mathfrak{A}X_1$ and $\Theta, \Lambda \subseteq \mathfrak{S}g^\mathfrak{A}X_2$. Then $a \in \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$ separates Γ from (Θ, Δ) if $\Gamma \rightarrow \{a\}$ and $\Theta \cup \{a\} \rightarrow \Lambda$. In this case we say that Γ can be separated from (Θ, Δ) . Otherwise Γ is inseparable from (Θ, Δ) with respect to \mathfrak{A} , X_1 , X_2 , or simply inseparable, when \mathfrak{A} , X_1 and X_2 are clear from context.

- (2) We say that $a \in \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$ separates Γ from Δ if a separates Γ from (\emptyset, Δ) , that is to say, if $\Gamma \rightarrow \{a\}$ and $\{a\} \rightarrow \Delta$.

Lemma 1.3. (*Essentially Ono's*) let G be the semigroup of all finite transformations on α . Let $\mathfrak{A} \in GPHA_\alpha$ and assume further that $\alpha \sim \Delta x$ is infinite for every x in A . Suppose that $\Gamma \subseteq \mathfrak{S}g^\mathfrak{A}X_1$ and $\Theta, \Lambda \subseteq \mathfrak{S}g^{\mathfrak{A}_2}X_2$. If Γ is inseparable from (Θ, Λ) with respect to X_1 and X_2 , then there exist an ω dilation \mathfrak{B} of \mathfrak{A} , $\Gamma' \subseteq \mathfrak{S}g^\mathfrak{B}X_1$ and $\Theta', \Delta' \subseteq \mathfrak{S}g^\mathfrak{B}X_2$, such that

1. $\Gamma \subseteq \Gamma'$ and Γ' is saturated in $\mathfrak{S}g^\mathfrak{B}X_1$.
2. $\Theta \subseteq \Theta'$ and $\Lambda \subseteq \Delta'$ and (Θ', Δ') is Henkin complete.
3. $(\Gamma' \cap \Theta', \Delta')$ is Henkin complete and $\Delta' = \Delta' \cap \mathfrak{S}g^\mathfrak{B}(X_1 \cap X_2)$.

4. Γ is inseparable from (Θ', Λ') .

Proof. For each $i \in \omega$, we will construct subsets Γ_i of $\mathfrak{S}g^{\mathfrak{B}}(X_1)$ and Θ_i, Λ_i of $\mathfrak{S}g^{\mathfrak{B}}(X_2)$ satisfying the following

$$(1) \quad \Gamma = \Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \dots,$$

$$\Theta = \Theta_0 \subseteq \Theta_1 \subseteq \Theta_2 \subseteq \dots,$$

$$\Lambda = \Lambda_0 \subseteq \Lambda_1 \subseteq \Lambda_2 \subseteq \dots$$

$$(2) \quad \Gamma_i \text{ is inseparable from } (\Theta_i, \Lambda_i) \text{ with respect to } \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2).$$

Let $(a_i : i \in \omega)$ be an enumeration of $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$, $(b_i : i \in \omega)$ be an enumeration of $\mathfrak{S}g^{\mathfrak{B}}(X_1) \sim \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ and $(c_i : i \in \omega)$ be an enumeration of $\mathfrak{S}g^{\mathfrak{B}}(X_2) \sim \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$.

First observe that Γ is inseparable from (Θ, Λ) with respect to the big algebra \mathfrak{B} and X_1 and X_2 . For if not, then there exists $b \in \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ such that $\Gamma \rightarrow \{b\}$ and $\Theta \cup \{b\} \rightarrow \Lambda$. By lemma ??, there is a finite set J , such that

$$c_{(J)}b \in \mathfrak{N}r_{\alpha}\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2) = \mathfrak{S}g^{\mathfrak{N}r_{\alpha}\mathfrak{B}}(X_1 \cap X_2) = \mathfrak{S}g^{\mathfrak{A}}(X_1 \cap X_2),$$

but then we would have $\Gamma \rightarrow \{c_{(J)}b\}$ and $\Theta \cup \{c_{(J)}b\} \rightarrow \Lambda$ which is impossible. So, (2) holds when $i = 0$. Next, suppose that $i > 0$ and inductively for each $k < i$, Γ_k, Θ_k and Λ_k satisfying the conditions (1), (2), are defined. We also assume inductively that

$$\beta \sim \bigcup_{x \in \Gamma_i} \Delta x \cup \bigcup_{x \in \Theta_i} \Delta x \cup \bigcup_{x \in \Lambda_i} \Delta x$$

is infinite for all i . This is clearly satisfied for the base of induction. We treat three cases separately.

1. $i = 3m + 1$. Since the sentence a_m belongs to $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ either $\Gamma_{i-1} \cup \{a_m\}$ is inseparable from $(\Theta_{i-1} \cup \{a_m, \Lambda_{i-1}\})$, or Γ_{i-1} is inseparable from $(\Theta_{i-1}, \Lambda_{i-1} \cup \{a_m\})$. Suppose that the former case holds.

We distinguish between two subcases:

Subcase (1): $a_m \neq c_k x$ for all $k < \beta$ and $x \in B$. Set $\Gamma_i = \Gamma_{i-1} \cup \{a_m\}$, $\Theta_i = \Theta_{i-1} \cup \{a_m\}$ and $\Lambda_i = \Lambda_{i-1}$.

Subcase (2): $a_m = c_k a$ for some $k < \beta$ and $a \in B$. Then set $\Gamma_i = \Gamma_{i-1} \cup \{a_m, s_j^k a\}$, $\Theta_i = \Theta_{i-1} \cup \{a_m, s_j^k a\}$ and $\Lambda_i = \Lambda_{i-1}$, where

$$j \in \beta \sim \bigcup_{x \in \Gamma_{i-1}} \Delta x \cup \bigcup_{x \in \Theta_{i-1}} \Delta x \cup \bigcup_{x \in \Lambda_{i-1}} \Delta x.$$

Such a j exists by the induction hypothesis. Then Γ_i is inseparable from (Θ_i, Λ_i) .

Suppose next that the latter case holds. Then, define $\Gamma_i = \Gamma_{i-1}$, $\Theta_i = \Theta_{i-1}$ and $\Lambda_i = \Lambda_{i-1} \cup \{a_m\}$.

In each subcase, as easily checked, the conditions (1) and (2) hold for Γ_i , Θ_i and Λ_i .

2. The case where $i = 3m+2$. Suppose first that some sentence in $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ that separates $\Gamma_{i-1} \cup \{b_m\}$ from $(\Theta_{i-1}, \Lambda_{i-1})$. Then, define $\Gamma_i = \Gamma_{i-1}$, $\Theta_i = \Theta_{i-1}$, $\Lambda_i = \Lambda_{i-1}$. Next suppose otherwise.

Subcase 1: When b_m is not of the form $c_k b$ for any $k \in \beta$ and $b \in B$; we define $\Gamma_i = \Gamma_{i-1} \cup \{b_m\}$, $\Theta_i = \Theta_{i-1}$ and $\Lambda_i = \Lambda_{i-1}$.

Subcase 2: When b_m is equal to $c_k b$ for some $k \in \beta$ and $b \in B$, define $\Gamma_i = \Gamma_{i-1} \cup \{b_m, s_j^k b\}$, $\Theta_i = \Theta_{i-1}$ and $\Lambda_i = \Lambda_{i-1}$ where

$$j \in \beta \sim \bigcup_{x \in \Gamma_{i-1}} \Delta x \cup \bigcup_{x \in \Theta_{i-1}} \Delta x \cup \bigcup_{x \in \Lambda_{i-1}} \Delta x.$$

In any case, we have Γ_i is inseparable from (Θ_i, Λ_i) .

3. The case where $i = 3m + 3$. As in the two previous cases, either Γ_{i-1} is inseparable from $(\Theta_{i-1} \cup \{c_m\}, \Lambda_{i-1})$ or Γ_{i-1} is inseparable from $(\Theta_{i-1}, \Lambda_{i-1} \cup \{c_m\})$. If it is the latter case, then set $\Gamma_i = \Gamma_{i-1}$, $\Theta_i = \Theta_{i-1}$ and $\Lambda_i = \Lambda_{i-1} \cup \{c_m\}$. If it is the former case, when c_m is not of the form $c_k c$ for any $k \in \beta$ and $c \in B$, define $\Gamma_i = \Gamma_{i-1}$, $\Theta_i = \Theta_{i-1} \cup \{c_m\}$ and $\Lambda_i = \Lambda_{i-1}$. If there exists $c \in \mathfrak{B}$, and $k \in \beta$ such that $c_m = c_k c$, set $\Gamma_i = \Gamma_{i-1}$, $\Theta_i = \Theta_{i-1} \cup \{c_m, s_j^k c\}$ and $\Lambda_i = \Lambda_{i-1}$ where

$$j \in \beta \sim \bigcup_{x \in \Gamma_{i-1}} \Delta x \cup \bigcup_{x \in \Theta_{i-1}} \Delta x \cup \bigcup_{x \in \Lambda_{i-1}} \Delta x.$$

In each case, the conditions (1) and (2) are satisfied.

Now, we define $\Gamma' = \bigcup_{i < \omega} \Gamma_i$, $\Theta' = \bigcup_{i < \omega} \Theta_i$ and $\Lambda' = \bigcup_{i < \omega} \Lambda_i$. Clearly, Γ' is inseparable from (Θ', Λ') , by (1) and (2).

We will prove that Γ' is saturated in $\mathfrak{S}g^{\mathfrak{B}}(X_1)$.

We first prove

Claim For any b in $\mathfrak{S}g^{\mathfrak{B}} X_1$, b is in Γ' if and only if $\Gamma_i \cup \{b\}$ is inseparable from (Θ_i, Λ_i) for every i .

Proof of Claim. Suppose that for each i , $\Gamma_i \cup \{b\}$ is inseparable from (Θ_i, Λ_i) . Assume that $b = b_m$ and let $k = 3m + 1$ if b belongs to $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$ and $k = 3m + 2$ otherwise. Then $b \in \Gamma_k \subseteq \Gamma'$, by the fact that $\Gamma_{k-1} \cup \{b_m\}$ is inseparable from $(\Theta_{k-1}, \Lambda_{k-1})$.

Conversely, suppose that $b \in \Gamma'$. Let $J = \{i \in \omega : b \in \Gamma_i\}$ and let $h = \min J$. Then for any i such that $h \leq i$, $\Gamma_i \cup \{b\}$ is inseparable from (Θ_i, Λ_i) , since $\Gamma_i \cup \{b\} = \Gamma_i$. So, suppose, seeking a contradiction that for some $i < h$, we have $\Gamma_i \cup \{b\}$ is separated from (Θ_i, Λ_i) . Then, it follows that $\Gamma_h (= \Gamma_h \cup \{b\})$ is also separated from (Θ_h, Λ_h) . But, this contradicts (2). Hence, $\Gamma_i \cup \{b\}$ is inseparable from (Θ_i, Λ_i) for each i . ■

Now, we will show that $(\Gamma', \mathfrak{S}g^{\mathfrak{B}}X_1 \sim \Gamma')$ is consistent. If not, then, there exist $a_1 \dots a_m$ in Γ' and $b_1 \dots b_n$ in $\mathfrak{S}g^{\mathfrak{B}}(X_1) \sim \Gamma$ such that $\bigwedge_{j=1}^m a_j \leq \bigvee_{j=1}^n b_j$. For each j , there exist a number k_j and an element $d_j \in \mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$, such that d_j separates $\Gamma_{k_j} \cup \{b_j\}$ from $(\Theta_{k_j}, \Lambda_{k_j})$. Take $k \in \omega$, such that $k_j < k$ for each j and $a_1 \dots a_m \in \Gamma_k$. Such a k of course exists. Now $\bigvee_{j=1}^n d_j$ in $\mathfrak{S}g^{\mathfrak{B}}(X_1 \cap X_2)$, separates $\Gamma_k \cup \{\bigvee_{j=1}^n b_j\}$ from (Θ_k, Λ_k) . Since $a_1 \dots a_m \in \Gamma_k$ and $\bigwedge_{j=1}^m a_j \leq \bigvee_{j=1}^n b_j$, then we can infer that $\bigvee_{j=1}^n d_j$ separates also Γ_k from (Θ_k, Λ_k) . But this contradicts (2).

Hence, $(\Gamma', \mathfrak{S}g^{\mathfrak{B}}X_1 \sim \Gamma')$ is consistent. By construction it is also Henkin-complete in $\mathfrak{S}g^{\mathfrak{B}}X_1$. Thus, Γ' is saturated in $\mathfrak{S}g^{\mathfrak{B}}(X_1)$. That (Θ', Λ') is Henkin-complete in $\mathfrak{S}g^{\mathfrak{B}}X_2$ is absolutely straightforward. From this, it immediately follows that $(\Gamma' \cap \Theta', \Delta')$ is consistent. Let $a \in \mathfrak{S}g^{\mathfrak{B}}(X_0 \cap X_1)$. Suppose that $a \notin \Delta'$. Then, $a \in \Theta'$, since (Θ', Λ') is complete in $\mathfrak{S}g^{\mathfrak{B}}X_1$. On the other hand, a is in either Γ' or Λ' by our construction. Hence, $a \in \Gamma'$ and, therefore, $a \in \Gamma' \cap \Theta'$. This means that $(\Gamma' \cap \Theta', \Delta')$ is complete, and also Henkin-complete by construction. ■

Remark 1.4. The proof works verbatim when G is strongly rich.

Theorem 1.5. Let $\mathfrak{A} = \mathfrak{F}r_{\mu}^{\rho}V$, where $|\mu| = \omega$, $\Gamma_0 \subseteq \mathfrak{S}g^{\mathfrak{A}}X_1$ and $\Theta_0, \Lambda_0 \subseteq \mathfrak{S}g^{\mathfrak{A}}X_2$. If Γ_0 is inseparable from (Θ_0, Λ_0) then the theory $(\Gamma_0 \cup \Theta_0, \Lambda_0)$ is satisfiable. That is to say, there exists $\mathfrak{K} = (K, \leq \{X_k\}_{k \in K} \{V_k\}_{k \in K})$, a homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{F}r_K$, $k_0 \in K$, and $x \in V_{k_0}$, such that for all $p \in \Gamma_0 \cup \Theta_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 1$ and for all $p \in \Lambda_0$ if $\psi(p) = (g_k)$, then $g_{k_0}(x) = 0$.

Proof. We provide the proof when G is the set of all finite transformations. Fix ρ such that $\alpha \sim \rho(i)$ is infinite for every $i \in \mu$. Form a sequence of minimal dilations $\mathfrak{A} = \mathfrak{B}_0 \subseteq \mathfrak{B}_1 \subseteq \dots \mathfrak{B}_{\infty}$, where for $n \leq \omega$, $\mathfrak{B}_n \in GPHA_{\alpha+n}$. Now by lemma ??, we have for $k < l \leq \omega$, $\mathfrak{B}_k = \mathfrak{N}r_k \mathfrak{B}_l$ and for all $X \subseteq B_k$, we have $\mathfrak{S}g^{\mathfrak{B}_k}X = \mathfrak{N}r_k \mathfrak{S}g^{\mathfrak{B}_l}X$. In this case we can give a more concrete description of the dilations. It will turn out, as we proceed to show in a minute, that \mathfrak{B}_l is the dimension restricted free algebra on μ generators in $\alpha + l$ dimensions restricted by ρ . For $n \leq \omega$, let $\beta = \alpha + n$. For brevity, we write $Hom(\mathfrak{A}, \mathfrak{B})$ for the set of all homomorphisms from \mathfrak{A} to \mathfrak{B} . Let $V_I = GPHA_I$. In view of lemma ??, it suffices to show that the sequence $\langle \eta / Cr_{\mu}^{\rho}V_{\beta} : \eta < \mu \rangle$ V_{α} - freely generates

$\mathfrak{Nr}_\alpha \mathfrak{Fr}_\mu^\rho(V_\beta)$. Towards this end, let $\mathfrak{B} \in V_\alpha$ and $a = \langle a_\eta : \eta < \mu \rangle \in {}^\mu B$ be such that $\Delta a_\eta \subseteq \rho(\eta)$ for all $\eta < \mu$. Since $\mathfrak{Fr}_\mu^\rho(V_\alpha)$ is in Dc_α , we have $|\alpha \setminus \bigcup_{\xi \in \Gamma} \rho\xi| \geq \omega$ for each finite $\Gamma \subseteq \mu$. Assuming that Rga generates \mathfrak{B} , we have $\mathfrak{B} \in Dc_\alpha$. Therefore $\mathfrak{B} \in S\mathfrak{Nr}_\alpha V_\beta$. Let $\mathfrak{D} = \mathfrak{Fr}_\mu^\rho(V_\beta)$.

We claim that $x = \langle \eta / Cr_\mu^\rho V_\beta : \eta < \mu \rangle \in {}^\mu D S\mathfrak{Nr}_\alpha K_\beta$ - freely generates $\mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{D}} Rgx$. Indeed, consider $\mathfrak{C} \in \mathfrak{Nr}_\alpha V_\beta$ and $y \in {}^\mu C$ such that $\Delta y_\eta \subseteq \rho\eta$ for all $\eta < \mu$. Let $\mathfrak{C}' \in V_\beta$ be such that $\mathfrak{C} = \mathfrak{Nr}_\alpha \mathfrak{C}'$. Then clearly $y \in {}^\mu C'$ and $\Delta y_\eta \subseteq \alpha$ for all $\eta < \mu$. Then there exists $h \in \text{Hom}(\mathfrak{D}, \mathfrak{C}')$ such that $h \circ x = y$. Hence $h \in \text{Hom}(\mathfrak{Nr}_\alpha \mathfrak{D}, \mathfrak{Nr}_\alpha \mathfrak{C}')$, thus $h \in \text{Hom}(\mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{D}} Rgx, \mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{C}'} h(Rgx))$. Since $Rgx \subseteq Nr_\alpha D$, we have $h \in \text{Hom}(\mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{D}} Rgx, \mathfrak{C})$. We have proved our claim.

Therefore there exists, by universal property of dimension restricted free algebras, a homomorphism $h : \mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{D}} Rgx \rightarrow \mathfrak{B}$ such that $h(\eta / Cr_\mu^\rho K_\beta) = a_\eta$. But $\mathfrak{Nr}_\alpha \mathfrak{Fr}_\mu^\rho(K_\beta) = \mathfrak{Nr}_\alpha (\mathfrak{Sg}^{\mathfrak{D}} Rgx) = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{D}} Rgx$. Therefore $\langle \eta / Cr_\mu^\rho(V_\beta) : \eta < \mu \rangle V_\alpha$ freely generates $\mathfrak{Nr}_\alpha \mathfrak{Fr}_\mu^\rho(V_\beta)$. We have proved that $\mathfrak{Fr}_\mu^\rho V_\alpha \cong \mathfrak{Nr}_\alpha \mathfrak{Fr}_\mu^\rho V_\beta$, via an isomorphism that fixes the generators. Hence the (minimal) dilations are nothing more than dimension restricted free algebras in extra dimensions $\leq \omega$, also restricted by ρ .

Now Γ_0 is inseparable from (Θ_0, Λ_0) with respect to \mathfrak{B}, X_1 and X_2 , and so there exist by lemma 1.3, $\Gamma'_0 \subseteq \mathfrak{Sg}^{\mathfrak{B}_1}(X_1)$, and $\Theta'_0, \Lambda'_0 \subseteq \mathfrak{Sg}^{\mathfrak{B}_1}(X_2)$ such that $(\Gamma'_0, \Sigma'_0), (\Theta'_0, \Lambda'_0)$ and $(\Gamma'_0 \cap \Theta'_0, \Delta'_0)$ are Henkin complete in $\mathfrak{Sg}^{\mathfrak{B}_1}(X_1)$, $\mathfrak{Sg}^{\mathfrak{B}_2}(X_2)$ and $\mathfrak{Sg}^{\mathfrak{B}}(X_1 \cap X_2)$ respectively, where $\Sigma'_0 = \mathfrak{Sg}^{\mathfrak{B}_1}(X_1) \sim \Gamma'_0$ and $\Delta'_0 = \Lambda'_0 \cap \mathfrak{Sg}^{\mathfrak{B}_1}(X_1 \cap X_2)$.

We define inductively for each $k \in \omega$ a set M_k each of whose members is a pair of the form (Γ, θ) such that $\Gamma \subseteq \mathfrak{Sg}^{\mathfrak{B}_k}(X_1)$, and $\Theta \subseteq \mathfrak{Sg}^{\mathfrak{B}_k}(X_2)$, and the following hold:

- (1) $\Gamma'_0 \subseteq \Gamma, \Theta'_0 \subseteq \Theta$,
- (2) $(\Gamma, \Sigma), (\Theta, \Lambda)$ and $(\Gamma \cap \Theta, \Sigma \cap \Lambda)$ are Henkin complete in $\mathfrak{Sg}^{\mathfrak{B}_k}(X_1), \mathfrak{Sg}^{\mathfrak{B}_k}(X_2)$ and $\mathfrak{Sg}^{\mathfrak{B}_k}(X_1 \cap X_2)$, respectively, where $\Sigma = \mathfrak{Sg}^{\mathfrak{B}_k}(X_1) \sim \Gamma$ and $\Lambda = \mathfrak{Sg}^{\mathfrak{B}_k}(X_2) \sim \Theta$.

1. At $k = 0$, set $W_0 = \{(\Gamma'_0, \Theta'_0)\}$.

2. Suppose that W_{k-1} has already been defined and that each member of W_{k-1} satisfies (1) and (2). For each $(\Gamma, \Theta) \in M_{k-1}$ for all $a \in \Sigma \cup \Lambda$ of the form $d \rightarrow e$ or $\mathbf{q}_i a$ we construct a pair satisfying (1) and (2) and put in into M_k . Here (a) and (b) are analogous to Claims 1 and 2 in the proof of ??.

First take $d \rightarrow e \in \Lambda$. Then Γ is inseparable from $(\Theta \cup \{d\}, e)$ with respect to $\mathfrak{Sg}^{B_{k-1}}(X_1 \cap X_2)$, so there exists $\Gamma' \subseteq \mathfrak{Sg}^{\mathfrak{B}_k}(X_1)$ and $\Theta', \Lambda' \subseteq \mathfrak{Sg}^{\mathfrak{B}_k}(X_2)$ satisfying (1) and (2), and $\Theta \cup \{d\} \subseteq \Theta'$ and $e \in \Lambda'$. If $\mathbf{q}_i a \in \Lambda$. Then Γ is inseparable from $(\Theta, \mathbf{s}_k^i a)$ where $k \in \text{dim} B_k \sim \text{dim} B_{k-1}$, with respect to $\mathfrak{Sg}^{B_{k-1}}(X_1 \cap X_2)$. Now let $K = \bigcup_{k \in \omega} W_k$; K is the set of worlds. If

$i = (\Delta, \Gamma) \in W_k$, let $M_k = \omega + k.n$. If $i_1 = (\Delta_1, \Gamma_1)$ and $i_2 = (\Delta_2, \Gamma_2)$ are in K , then set

$$i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2.$$

This is a preorder on K ; define the kripke system \mathfrak{K} based on the set of worlds K as before. Set $\psi : \mathfrak{A} \rightarrow \mathfrak{F}_{\mathfrak{K}}$ by $\psi_1(p) = (f_l)$ such that if $l = (\Delta, \Gamma) \in K$ is saturated in \mathfrak{A}_k , and $M_k = \dim \mathfrak{A}_k$, then for $x \in V_k = \bigcup_{p \in G_n} {}^\alpha M_k^{(p)}$,

$$f_k(x) = 1 \iff s_{x \cup (Id_{M_k} \sim \alpha)}^{\mathfrak{A}_n} p \in \Delta.$$

Let $k_0 = (\Delta_\omega, \Gamma_\omega)$ be provided by lemma 1.3, that is a Henkin complete saturated extension of (Δ_0, Γ_0) in \mathfrak{B}_1 , then ψ , k_0 and Id are as desired. ■

Remark 1.6. The proof works verbatim when G is strongly rich. Theorem is the special case of MAIN proved in part 1, when the first pair is just (Θ, \perp) . This weaker version suffices to prove interpolation, see the proof of theorem ? in part 3.

2 Presence of diagonal elements

All results, in Part 1, up to the previous theorem, are proved in the absence of diagonal elements. Now let's see how far we can go if we have diagonal elements. Considering diagonal elements, as we shall see, turn out to be problematic but not hopeless.

Our representation theorem has to respect diagonal elements, and this seems to be an impossible task with the presence of infinitary substitutions, unless we make a compromise that is, from our point of view, acceptable. The interaction of substitutions based on infinitary transformations, together with the existence of diagonal elements tends to make matters 'blow up'; indeed this even happens in the classical case, when the class of (ordinary) set algebras ceases to be closed under ultraproducts [44]. The natural thing to do is to avoid those infinitary substitutions at the start, while finding the interpolant possibly using such substitutions. We shall also show that in some cases the interpolant has to use infinitary substitutions, even if the original implication uses only finite transformations.

So for an algebra \mathfrak{A} , we let $\mathfrak{A} \text{red}$ denote its reduct when we discard infinitary substitutions. $\mathfrak{A} \text{red}$ satisfies cylindric algebra axioms.

Theorem 2.1. *Let α be an infinite set. Let G be a semigroup on α containing at least one infinitary transformation. Let $\mathfrak{A} \in GPHAE_\alpha$ be the free G algebra generated by X , and suppose that $X = X_1 \cup X_2$. Let (Δ_0, Γ_0) , (Θ_0, Γ_0^*) be two consistent theories in $\mathfrak{S}g^{\mathfrak{A} \text{red}} X_1$ and $\mathfrak{S}g^{\mathfrak{A} \text{red}} X_2$, respectively. Assume that*

$\Gamma_0 \subseteq \mathfrak{S}g^{\mathfrak{A}}(X_1 \cap X_2)$ and $\Gamma_0 \subseteq \Gamma_0^*$. Assume, further, that $(\Delta_0 \cap \Theta_0 \cap \mathfrak{S}g^{\mathfrak{A}}X_1 \cap \mathfrak{S}g^{\mathfrak{A}}X_2, \Gamma_0)$ is complete in $\mathfrak{S}g^{\mathfrak{A}}X_1 \cap \mathfrak{S}g^{\mathfrak{A}}X_2$. Then there exist $\mathfrak{K} = (K, \leq, \{X_k\}_{k \in K}, \{V_k\}_{k \in K})$, a homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{F}_K$, $k_0 \in K$, and $x \in V_{k_0}$, such that for all $p \in \Delta_0 \cup \Theta_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 1$ and for all $p \in \Gamma_0^*$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 0$.

Proof. The first half of the proof is almost identical to that of lemma ?? . We highlight the main steps, for the convenience of the reader, except that we only deal with the case when G is strongly rich. Assume, as usual, that α , G , \mathfrak{A} and X_1 , X_2 , and everything else in the hypothesis are given. Let I be a set such that $\beta = |I \sim \alpha| = \max(|A|, |\alpha|)$. Let $(K_n : n \in \omega)$ be a family of pairwise disjoint sets such that $|K_n| = \beta$. Define a sequence of algebras $\mathfrak{A} = \mathfrak{A}_0 \subseteq \mathfrak{A}_1 \subseteq \mathfrak{A}_2 \subseteq \mathfrak{A}_3 \dots \subseteq \mathfrak{A}_n \dots$ such that \mathfrak{A}_{n+1} is a minimal dilation of \mathfrak{A}_n and $\dim(\mathfrak{A}_{n+1}) = \dim \mathfrak{A}_n \cup K_n$. We denote $\dim(\mathfrak{A}_n)$ by I_n for $n \geq 1$. The proofs of Claims 1 and 2 in the proof of ?? are the same.

Now we prove the theorem when G is a strongly rich semigroup. Let

$$K = \{((\Delta, \Gamma), (T, F)) : \exists n \in \omega \text{ such that } (\Delta, \Gamma), (T, F)$$

is a matched pair of saturated theories in $\mathfrak{S}g^{\mathfrak{A}_n}X_1, \mathfrak{S}g^{\mathfrak{A}_n}X_2\}$.

We have $((\Delta_0, \Gamma_0), (\Theta_0, \Gamma_0^*))$ is a matched pair but the theories are not saturated. But by lemma ?? there are $T_1 = (\Delta_\omega, \Gamma_\omega)$, $T_2 = (\Theta_\omega, \Gamma_\omega^*)$ extending (Δ_0, Γ_0) , (Θ_0, Γ_0^*) , such that T_1 and T_2 are saturated in $\mathfrak{S}g^{\mathfrak{A}_1}X_1$ and $\mathfrak{S}g^{\mathfrak{A}_1}X_2$, respectively. Let $k_0 = ((\Delta_\omega, \Gamma_\omega), (\Theta_\omega, \Gamma_\omega^*))$. Then $k_0 \in K$, and k_0 will be the desired world and x will be specified later; in fact x will be the identity map on some specified domain.

If $i = ((\Delta, \Gamma), (T, F))$ is a matched pair of saturated theories in $\mathfrak{S}g^{\mathfrak{A}_n}X_1$ and $\mathfrak{S}g^{\mathfrak{A}_n}X_2$, let $M_i = \dim \mathfrak{A}_n$, where n is the least such number, so n is unique to i . Let

$$\mathbf{K} = (K, \leq, \{M_i\}, \{V_i\})_{i \in \mathfrak{K}},$$

where $V_i = \bigcup_{p \in G_n, p \text{ a finitary transformation}} {}^\alpha M_i^{(p)}$ (here we are considering only substitutions that move only finitely many points), and G_n is the strongly rich semigroup determining the similarity type of \mathfrak{A}_n , with n the least number such i is a saturated matched pair in \mathfrak{A}_n , and \leq is defined as follows: If $i_1 = ((\Delta_1, \Gamma_1), (T_1, F_1))$ and $i_2 = ((\Delta_2, \Gamma_2), (T_2, F_2))$ are in \mathbf{K} , then set

$$i_1 \leq i_2 \iff M_{i_1} \subseteq M_{i_2}, \Delta_1 \subseteq \Delta_2, T_1 \subseteq T_2.$$

We are not yet there, to preserve diagonal elements we have to factor out \mathbf{K} by an infinite family equivalence relations, each defined on the dimension of \mathfrak{A}_n , for some n , which will actually turn out to be a congruence in an exact sense. As usual, using freeness of \mathfrak{A} , we will define two maps on $\mathfrak{A}_1 = \mathfrak{S}g^{\mathfrak{A}_1}X_1$ and

$\mathfrak{A}_2 = \mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_2$, respectively; then those will be pasted to give the required single homomorphism.

Let $i = ((\Delta, \Gamma), (T, F))$ be a matched pair of saturated theories in $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_1$ and $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_2$, let $M_i = \dim \mathfrak{A}_n$, where n is the least such number, so n is unique to i . For $k, l \in \dim \mathfrak{A}_n = I_n$, set $k \sim_i l$ iff $\mathbf{d}_{kl}^{\mathfrak{A}_n} \in \Delta \cup T$. This is well defined since $\Delta \cup T \subseteq \mathfrak{A}_n$. We omit the superscript \mathfrak{A}_n . These are infinitely many relations, one for each i , defined on I_n , with n depending uniquely on i , we denote them uniformly by \sim to avoid complicated unnecessary notation. We hope that no confusion is likely to ensue. We claim that \sim is an equivalence relation on I_n . Indeed, \sim is reflexive because $\mathbf{d}_{ii} = 1$ and symmetric because $\mathbf{d}_{ij} = \mathbf{d}_{ji}$; finally E is transitive because for $k, l, u < \alpha$, with $l \notin \{k, u\}$, we have

$$\mathbf{d}_{kl} \cdot \mathbf{d}_{lu} \leq \mathbf{c}_l(\mathbf{d}_{kl} \cdot \mathbf{d}_{lu}) = \mathbf{d}_{ku},$$

and we can assume that $T \cup \Delta$ is closed upwards. For $\sigma, \tau \in V_k$, define $\sigma \sim \tau$ iff $\sigma(i) \sim \tau(i)$ for all $i \in \alpha$. Then clearly \sim is an equivalence relation on V_k .

Let $W_k = V_k / \sim$, and $\mathfrak{K} = (K, \leq, M_k, W_k)_{k \in K}$, with \leq defined on K as above. We write $h = [x]$ for $x \in V_k$ if $x(i) / \sim = h(i)$ for all $i \in \alpha$; of course X may not be unique, but this will not matter. Let $\mathfrak{F}_{\mathfrak{K}}$ be the set algebra based on the new Kripke system \mathfrak{K} obtained by factoring out \mathbf{K} .

Set $\psi_1 : \mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_1 \rightarrow \mathfrak{F}_{\mathfrak{K}}$ by $\psi_1(p) = (f_k)$ such that if $k = ((\Delta, \Gamma), (T, F)) \in K$ is a matched pair of saturated theories in $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_1$ and $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}}X_2$, and $M_k = \dim \mathfrak{A}_n$, with n unique to k , then for $x \in W_k$

$$f_k([x]) = 1 \iff \mathbf{s}_{x \cup (Id_{M_k \sim \alpha})}^{\mathfrak{A}_n} p \in \Delta \cup T,$$

with $x \in V_k$ and $[x] \in W_k$ is define as above.

To avoid cumbersome notation, we write $\mathbf{s}_x^{\mathfrak{A}_n} p$, or even simply $\mathbf{s}_x p$, for $\mathbf{s}_{x \cup (Id_{M_k \sim \alpha})}^{\mathfrak{A}_n} p$. No ambiguity should arise because the dimension n will be clear from context.

We need to check that ψ_1 is well defined. It suffices to show that if $\sigma, \tau \in V_k$ if $\sigma \sim \tau$ and $p \in \mathfrak{A}_n$, with n unique to k , then

$$\mathbf{s}_{\tau} p \in \Delta \cup T \text{ iff } \mathbf{s}_{\sigma} p \in \Delta \cup T.$$

This can be proved by induction on the cardinality of $J = \{i \in I_n : \sigma i \neq \tau i\}$, which is finite since we are only taking finite substitutions. If J is empty, the result is obvious. Otherwise assume that $k \in J$. We recall the following piece of notation. For $\eta \in V_k$ and $k, l < \alpha$, write $\eta(k \mapsto l)$ for the $\eta' \in V$ that is the same as η except that $\eta'(k) = l$. Now take any

$$\lambda \in \{\eta \in I_n : \sigma^{-1}\{\eta\} = \tau^{-1}\{\eta\} = \{\eta\}\} \setminus \Delta x.$$

This λ exists, because σ and τ are finite transformations and \mathfrak{A}_n is a dilation with enough spare dimensions. We have by cylindric axioms (a)

$$s_\sigma x = s_{\sigma k}^\lambda s_{\sigma(k \mapsto \lambda)} p.$$

We also have (b)

$$s_{\tau k}^\lambda (d_{\lambda, \sigma k} \wedge s_\sigma p) = d_{\tau k, \sigma k} s_\sigma p,$$

and (c)

$$\begin{aligned} s_{\tau k}^\lambda (d_{\lambda, \sigma k} \wedge s_{\sigma(k \mapsto \lambda)} p) \\ = d_{\tau k, \sigma k} \wedge s_{\sigma(k \mapsto \tau k)} p. \end{aligned}$$

and (d)

$$d_{\lambda, \sigma k} \wedge s_{\sigma k}^\lambda s_{\sigma(k \mapsto \lambda)} p = d_{\lambda, \sigma k} \wedge s_{\sigma(k \mapsto \lambda)} p$$

Then by (b), (a), (d) and (c), we get,

$$\begin{aligned} d_{\tau k, \sigma k} \wedge s_\sigma p &= s_{\tau k}^\lambda (d_{\lambda, \sigma k} \cdot s_\sigma p) \\ &= s_{\tau k}^\lambda (d_{\lambda, \sigma k} \wedge s_{\sigma k}^\lambda s_{\sigma(k \mapsto \lambda)} p) \\ &= s_{\tau k}^\lambda (d_{\lambda, \sigma k} \wedge s_{\sigma(k \mapsto \lambda)} p) \\ &= d_{\tau k, \sigma k} \wedge s_{\sigma(k \mapsto \tau k)} p. \end{aligned}$$

The conclusion follows from the induction hypothesis. Now ψ_1 respects all quasipolyadic equality operations, that is finite substitutions (with the proof as before; recall that we only have finite substitutions since we are considering $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}} X_1$) except possibly for diagonal elements. We check those:

Recall that for a concrete Kripke frame $\mathfrak{F}_\mathbf{W}$ based on $\mathbf{W} = (W, \leq, V_k, W_k)$, we have the concrete diagonal element d_{ij} is given by the tuple $(g_k : k \in K)$ such that for $y \in V_k$, $g_k(y) = 1$ iff $y(i) = y(j)$.

Now for the abstract diagonal element in \mathfrak{A} , we have $\psi_1(d_{ij}) = (f_k : k \in K)$, such that if $k = ((\Delta, \Gamma), (T, F))$ is a matched pair of saturated theories in $\mathfrak{S}g^{\mathfrak{A}\mathfrak{A}_n} X_1, \mathfrak{S}g^{\mathfrak{A}\mathfrak{A}_n} X_2$, with n unique to i , we have $f_k([x]) = 1$ iff $s_x d_{ij} \in \Delta \cup T$ (this is well defined $\Delta \cup T \subseteq \mathfrak{A}_n$).

But the latter is equivalent to $d_{x(i), x(j)} \in \Delta \cup T$, which in turn is equivalent to $x(i) \sim x(j)$, that is $[x](i) = [x](j)$, and so $(f_k) \in d_{ij}^{\mathfrak{F}_\mathbf{R}}$. The reverse implication is the same.

We can safely assume that $X_1 \cup X_2 = X$ generates \mathfrak{A} . Let $\psi = \psi_1 \cup \psi_2 \upharpoonright X$. Then ψ is a function since, by definition, ψ_1 and ψ_2 agree on $X_1 \cap X_2$. Now by freeness ψ extends to a homomorphism, which we denote also by ψ from \mathfrak{A} into $\mathfrak{F}_\mathbf{R}$. And we are done, as usual, by ψ , k_0 and $Id \in V_{k_0}$. ■

Theorem ??, generalizes as is, to the expanded structures by diagonal elements. That is to say, we have:

Theorem 2.2. *Let G be the semigroup of finite transformations on an infinite set α and let δ be a cardinal > 0 . Let $\rho \in {}^\delta \wp(\alpha)$ be such that $\alpha \sim \rho(i)$ is infinite for every $i \in \delta$. Let \mathfrak{A} be the free G algebra with equality generated by X restricted by ρ ; that is $\mathfrak{A} = \mathfrak{F}\mathfrak{r}_\delta^\rho GPHA_\alpha$, and suppose that $X = X_1 \cup X_2$. Let (Δ_0, Γ_0) , (Θ_0, Γ_0^*) be two consistent theories in $\mathfrak{S}g^\mathfrak{A}X_1$ and $\mathfrak{S}g^\mathfrak{A}X_2$, respectively. Assume that $\Gamma_0 \subseteq \mathfrak{S}g^\mathfrak{A}(X_1 \cap X_2)$ and $\Gamma_0 \subseteq \Gamma_0^*$. Assume, further, that $(\Delta_0 \cap \Theta_0 \cap \mathfrak{S}g^\mathfrak{A}X_1 \cap \mathfrak{S}g^\mathfrak{A}X_2, \Gamma_0)$ is complete in $\mathfrak{S}g^\mathfrak{A}X_1 \cap \mathfrak{S}g^\mathfrak{A}X_2$. Then there exist a Kripke system $\mathfrak{K} = (K, \leq \{X_k\}_{k \in K} \{V_k\}_{k \in K})$, a homomorphism $\psi : \mathfrak{A} \rightarrow \mathfrak{F}_K$, $k_0 \in K$, and $x \in V_{k_0}$, such that for all $p \in \Delta_0 \cup \Theta_0$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 1$ and for all $p \in \Gamma_0^*$ if $\psi(p) = (f_k)$, then $f_{k_0}(x) = 0$.*

Proof. $\mathfrak{Rd}\mathfrak{A}$ is just \mathfrak{A} . ■

Remark 2.3. Theorem extends to the presence of diagonal elements by using a simpler version of the congruence relation defined above. For $n \in \omega$, and a pair $i = (\Delta, \Gamma)$ in \mathfrak{A}_n , for $k, l \in \dim \mathfrak{A}_n$, set $k \sim_i l$ iff $d_{kl}^{2n} \in \Delta$.

2.1 Neat Embeddings and the finitizability problem for intuitionistic predicate logic

Let α be an infinite ordinal. We set:

$$RGA_\alpha = \mathbf{SP}\{\mathfrak{F}_K : \mathbf{K} \text{ a Kripke system of dimension } \alpha\}.$$

We refer to RGA_α as the class of representable G algebras. The next theorem is analogous to the celebrated so-called neat embedding theorem of Henkin in cylindric algebras. For $\alpha < \beta$, let $\mathbf{S}\mathfrak{Nr}_\alpha G_\beta PHA_\beta = \{\mathfrak{A} \in G_\alpha PHA_\alpha : \exists \mathfrak{B} \in G_\beta PHA_\beta : \mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}\}$.

Now using the previous lemmas, we prove a neat embedding theorem:

Theorem 2.4. (1) *When G is strongly rich (and α is countable) or G consists of all transformations, then $RGA_\alpha = GPHA_\alpha$*

(2) *When G consists only of finite transformations, then we have $RGA_\alpha = \mathbf{S}\mathfrak{Nr}_\alpha G_{\alpha+\omega} PHA_{\alpha+\omega}$. In particular, RGA_α is a variety.*

Proof. First it is easy, but tedious, to verify soundness, namely, that $RGA_\alpha \subseteq GPHA_\alpha$. One just has to check that equational axioms for G algebras hold in set algebras based on Kripke systems. The other inclusion of (i) follows from ??, since for a class $K \subseteq GPHA_\alpha$, we have $\mathfrak{A} \in \mathbf{SP}K$, if and only if, for all non-zero $a \in \mathfrak{A}$, there exists $\mathfrak{B} \in K$ and a homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $f(a) \neq 0$.

We now show (ii). It suffices to show that for infinite ordinals $\alpha < \beta$, if \mathbf{K} is a Kripke system of dimension α and \mathbf{M} is one of dimension β , then

$\mathfrak{F}_K \subseteq \mathfrak{Nr}_\alpha \mathfrak{F}_M$. Assume that $\mathfrak{F}_K = \{f = (f_k : k \in K); f_k : {}^\alpha X^{(Id)} \rightarrow \mathfrak{D}, k \leq k' \implies f_k \leq f_{k'}\}$ and that $\mathfrak{F}_M = \{g = (g_k : k \in K); g_k : {}^\beta X^{(Id)} \rightarrow \mathfrak{D}, k \leq k' \implies g_k \leq g_{k'}\}$ are Kripke set algebras based on the given Kripke systems of dimension α and β , respectively, with operations defined as usual. Define $\Psi : \mathfrak{F}_K \rightarrow \mathfrak{F}_M$ by $(\psi f)_k = h_k$ where $h_k \upharpoonright \alpha = f_k$ and otherwise is equal to the zero element. Then Ψ neatly embeds \mathfrak{F}_K into \mathfrak{F}_M , that is it embeds \mathfrak{F}_K into $\mathfrak{Nr}_\alpha \mathfrak{F}_M$. This shows that $RGA_\alpha \subseteq S\mathfrak{Nr}_\alpha G_{\alpha+\omega} PHA_{\alpha+\omega}$.

The converse inclusion (which is basically an algebraic version of a completeness theorem) is slightly more difficult. Assume that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$. Let $\mathfrak{B}' = \mathfrak{S}g^\mathfrak{B} A$. Then $\mathfrak{B}' \in G_{\alpha+\omega} PHA_{\alpha+\omega}$. This follows from the following reasoning. Let $\beta = \alpha + \omega$. Then $|\beta \sim \alpha| \geq \omega$, and $\Delta a \subseteq \alpha$ for all $a \in A$, it follows by a simple inductive argument that for all $y \in \mathfrak{S}g^\mathfrak{B} A$, $|\beta \sim \Delta y| \geq \omega$. Hence by ??, \mathfrak{B}' is representable, and so is \mathfrak{A} , for $\mathfrak{Nr}_\alpha \mathfrak{B}'$ is representable and a subalgebra of a representable algebra is representable. To show that $\mathfrak{Nr}_\alpha \mathfrak{B}'$ is representable, it suffices to take $\mathfrak{F}_K = \{f = (f_k : k \in K); f_k : {}^\beta X_k^{(Id)} \rightarrow \mathfrak{D}, k \leq k' \implies f_k \leq f_{k'}\}$ and show that $\mathfrak{Nr}_\alpha \mathfrak{F}_K$ is representable. But clearly the latter is isomorphic to $\mathfrak{F}_K^\alpha = \{(f_k \upharpoonright \alpha : k \in K) : f_k \upharpoonright \alpha : {}^\alpha X_k^{(Id)} \rightarrow \mathfrak{D}, k \leq k', f_k \leq f_{k'}\}$.

Now we prove something stronger than what is required at the end of item (ii). We show that for infinite ordinals $\alpha < \beta$, $S\mathfrak{Nr}_\alpha G_\beta PHA_\beta$ is a variety. It is closed under forming subalgebras by definition. It is closed under products, since for any system of algebras $(\mathfrak{B}_i : i \in I)$ we have $\prod_{i \in I} \mathfrak{Nr}_\alpha \mathfrak{B}_i = \mathfrak{Nr}_\alpha \prod_{i \in I} \mathfrak{B}_i$, via the map $(a_i : i \in I) \mapsto (a_i : i \in I)$ which is evidently one to one, surjective and a homomorphism.

We show that the $S\mathfrak{Nr}_\alpha G_\beta PHA_\beta$ is also closed under homomorphic images. Let $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha \mathfrak{B}$, and I an ideal of \mathfrak{A} . We will show that that homomorphic image \mathfrak{A}/I of \mathfrak{A} is in $S\mathfrak{Nr}_\alpha G_\beta PHA_\beta$. Let J be the ideal generated by I in \mathfrak{B} . Then $J \cap A = I$. Define $f : \mathfrak{A}/I \rightarrow \mathfrak{Nr}_\alpha(\mathfrak{A}/J)$ by $f(x/I) = x/J$, then f is an embedding, and we are done. ■

2.2 The Finitizability for Heyting Polyadic algebras with and without equality

Item (1) in theorem 2.4 says that there is a finite schema axiomatizable variety consisting of subdirect products of set algebras based on Kripke systems (models) of a natural extension of ordinary predicate intuitionistic logic. This schema can be strictly cut to be a finite axiomatization, that can be easily implemented, using the methods of Sain in [44] where an analogous result addressing first order (classical) logic without equality is proved.

The following theorem is the algebraic version of weak completeness see theorem ? in part 3. The theorem for classical algebras and its logical consequences are proved in [44]. Almost the same proof works in our intuitionistic

context (note that (i) and (ii)) are obvious. We omit the proof, which can be easily recovered from [44] using fairly standard machinery of algebraic logic dealing with the “bridge” between universal algebraic results and their logical counterparts.

In the presence of diagonal elements we have:

Theorem 2.5. *Let G be rich, then we have*

- (i) $GPHAE_\omega$ consists of subreducts of full polyadic equality Heyting algebras.
- (ii) $\mathbf{SRd}_{ca}GPHAE_\omega = RGA_\omega$. In other words, the class of representable algebras of dimension ω coincides with the class of subreducts of $GPHAE_\omega$
- (iii) $\mathbf{H}GPHAE_\omega$ is a finitely based variety.

Proof. [44]. ■

Theorem 2.6. *For rich G , $GPHAE_\alpha$ is not closed under ultraproducts*

In the absence of diagonal elements, a stronger form of the above theorem holds, for in this case $GPHE_\omega$ is itself is a finitely based variety. In fact in this case we have:

Theorem 2.7. *Let G be strongly rich, then we have*

- (i) $GPHA_\omega$ consists of subreducts of full polyadic Heyting algebras.
- (ii) $\mathbf{SRd}_{ca}GPHAE_\omega = RGH_\omega$. In other words, the class of representable algebras of dimension ω coincides with the class of subreducts of $GPHAE_\omega$
- (iii) RGH_ω is a finitely based variety that has SUPAP.

Proof. [44]. ■

In a forthcoming publication for different semigroups G , we introduce an intuitionistic logic (with equality) $\mathfrak{L}_G(\mathfrak{L}_G^-)$, where formulas have infinite length extending predicate intuitionistic logic.

We show using our algebraic results that several such logics are complete and have the interpolation properties, while others do not, pending on G . In the presence of equality, our positive results are weaker.

When we deal with those logics whose atomic formulas exhaust the set of all variables; no variables lie outside formulas, we discover that the borderline, that turns negative results to positive ones are the presence of infinitary transformations, at least two, where one is injective and not onto and the other is its left inverse. These transformations move infinitely many variables, and permits one to form dilations in spare dimensions, an essential feature in forming

successive extensions of theories. We show that the presence of such transformations is also necessary, in the sense that their presence cannot be dispensed with. If only finite transformations are available, then, without imposing any conditions on dimension sets of elements, representability and amalgamation fail in a strong sense.

References

- [1] Andr  ka, H. *Complexity of equations valid in algebras of relations*. Annals of Pure and Applied logic, **89** (1997), p. 149 - 209.
- [2] Andreka, H., Nemeti I., Sayed Ahmed T. *A non representable quasi-polyadic algebra with a representable cylindric reduct* Studia Math Hungarica, in press
- [3] Sayed Ahmed, T. *On Amalgamation of Reducts of Polyadic Algebras*. Algebra Universalis **51** (2004), p.301-359.
- [4] Sayed Ahmed, T. *Algebraic Logic, where does it stand today?* Bulletin of Symbolic Logic, **11**(4) (2005), p. 465–516
- [5] Sayed Ahmed T, *Some results about neat reducts* Algebra Universalis, **1**(2010) p. 17-36.
- [6] Sayed Ahmed , T. *The class of polyadic algebras has the superamalgamation property* Mathematical Logic Quarterly **56**(1)(2010)p.103-112
- [7] Sayed Ahmed T. *The amalgamation property, and a problem of Henkin Monk and Tarski* Journal of Algebra, number theory, advances and applications **1**(2)(2009) p. 127-141
- [8] Sayed Ahmed, T. *On neat embeddings of cylindric algebras* Mathematical Logic Quarterly **55**(6)(2009)p.666-668
- [9] Sayed Ahmed *Classes of algebras without the amalgamation property* Logic Journal of IGPL. **1** (2011) 87-104.
- [10] Sayed Ahmed T. *Amalgamation of polyadic Heyting algebras* Studia Math Hungarica, in press.
- [11] Daigneault, A., *Freedom in polyadic algebras and two theorems of Beth and Craig*. Michigan Math. J.**11** (1963), p. 129-135.
- [12] Daigneault, A., and Monk,J.D., *Representation Theory for Polyadic algebras*. Fund. Math. **52**(1963) p.151-176.

- [13] Ferenczi, M., *On representation of neatly embeddable cylindric algebras* Journal of Applied Non-classical Logics, **10**(3-4) (2000) p.34-56
- [14] Ferenczi, M., *Finitary polyadic algebras from cylindric algebras*. Studia Logica **87**(1)(2007) p.1-11
- [15] Ferenczi, M., *On cylindric algebras satisfying the merry-go-round properties* Logic Journal of IGPL, **15**(2) (2007), p. 183-199
- [16] Ferenczi, M., *On representability of neatly embeddable cylindric algebras* Journal of Appl. Non-classical Logic, 3-4, 10 (2000) 1-11
- [17] Ferenczi M, *The polyadic representation* Transactions of Amer Math Society, to appear.
- [18] Gabbay M.D., Maksimova L. *Interpolation and Definability: Modal and Intuitionistic Logic* Oxford Science Publications (2005)
- [19] Gentzen, G., 1934-5, "Untersuchungen ber das logische Schliessen," Math. Zeitschrift 39: 176-210, 405-431.
- [20] Georgescu G. *A representation theorem for polyadic Heyting algebras* Algebra Universalis **14** (1982), 197-209.
- [21] Gdel, K., 1933, "Zur intuitionistischen Arithmetik und Zahlentheorie," Ergebnisse eines mathematischen Kolloquiums 4: 34-38.
- [22] Halmos, P., *Algebraic Logic*. Chelsea Publishing Co., New York, (1962.)
- [23] Henkin, L., *An extension of the Craig-Lyndon interpolation theorem* Journal of Symbolic Logic 28(3) (1963) p.201-216
- [24] Henkin, L., Monk, J.D., and Tarski, A., *Cylindric Algebras Part I*. North Holland, 1971.
- [25] Henkin, L., Monk, J.D., and Tarski, A., *Cylindric Algebras Part II*. North Holland, 1985.
- [26] Herrlich H, Strecker G. *Category theory* Allyn and Bacon, Inc, Boston (1973)
- [27] Heyting, A., *Die formalen Regeln der intuitionistischen Logik, in three parts*, Sitzungsber. preuss. Akad. Wiss.: 42-71,(1930) 158-169. English translation of Part I in Mancosu 1998: 311-327.
- [28] Heyting, A., 1956, *Intuitionism: An Introduction*, North-Holland Publishing, Amsterdam. Third Revised Edition (1971).

- [29] Hodges, W. *A shorter Model Theory*. Cambridge. University Press. 1997.
- [30] Johnson, J.S. *Amalgamation of Polyadic Algebras*. Transactions of the American Mathematical Society, **149**(1970) p.627-652
- [31] Pigozzi, D. *Amalgamation, congruence extension, and interpolation properties in algebras*. Algebra Universalis. **1**(1971), p.269-349.
- [32]
- [33] Keisler H.J., *A complete first order logic with infinitary predicates* Fund. Math **52**(1963) p.177-203
- [34] Kleene, S. C., 1952, *Introduction to Metamathematics*, Van Nostrand, Princeton.
- [35] Kripke "Semantical analysis of intuitionistic logic," in J. Crossley and M. A. E. Dummett, eds., 1965: 92-130.
- [36] Madárasz J. and Sayed Ahmed T., *Amalgamation, interpolation and epimorphisms*. Algebra Universalis **56** (2) (2007) p. 179-210.
- [37] Madárasz J. and Sayed Ahmed T. *Neat reducts and amalgamation in retrospect, a survey of results and some methods. Part 1: Results on neat reducts* Logic Journal of IGPL **17**(4)(2009) p.429-483
- [38] Madárasz J. and Sayed Ahmed T., *Neat reducts and amalgamation in retrospect, a survey of results and some methods. Part 2: Results on amalgamation* Logic Journal of IGPL (2009) doi: 10.1093/jigpal/jzp013
- [39] Monk J.D. *Polyadic Heyting algebras* Notices Amer Math Soc (1966) 735.
- [40] Maksimova, L. *Amalgamation and interpolation in normal modal logics*. Studia Logica **50**(1991) p.457-471.
- [41] Németi, I., Sági, G. *On the equational theory of representable polyadic algebras*. Journal of Symbolic Logic (65)(3) 2000, p. 1143-1167
- [42] Sagi, G, Ferenszi, M, *On some developments in the representation theory of cylindric-like algebras* Algebra Universalis, **55**(2-3)(2006) p.345-353

- [43] Sagi, G, Shelah S., *Weak and strong interpolation for algebraic logics*. Journal of Symbolic Logic **71**(2006) p.104-118.
- [44] Sain, I. *Searching for a finitizable algebraization of first order logic*. 8, Logic Journal of IGPL. Oxford University, Press (2000) no 4, p.495–589.
- [45]
- [46] A. Tarski, *Grundzuge der Systemenkalkuls. Erster Teil*. Fundamenta Mathematica, Vol. **25**, (1935), p.503-526. English translation in [A. Tarski, Logic, Semantics, Metamathematics. Papers from 1923 to 1938, edited by J. Corcoran, Hackett Pub. Co., Indianapolis, Indiana, second edition, (1983)]: Foundations of the calculus of systems, p.342-383.